Analytical Approach to a Slowly Deforming Channel Flow with Weak Permeability

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In this paper, we develop the analytical solution of the Navier-Stokes equations for a semi-infinite rectangular channel with porous and uniformly expanding or contracting walls by employing the homotopy perturbation method (HPM). The series solution of the governing problem is obtained. Some examples have been included. The results so obtained are compared with the existing literature and a remarkable improvement leads to an excellent agreement with the numerical results.

Key words: Homotopy Perturbation Method; Analytical Solution; Deformable Channel; Viscous Fluid; Permeable Walls.

1. Introduction

The history of such flow is well documented and presented in [1]. Another very important and interesting application of this analysis in the context of biomechanics is given by Majdalani et al. [2]. In view of the engineering applications, studies of laminar flow within permeable walls have recently received considerable attention [3 – 5]. We have revisited this problem both numerically and analytically. Numerical results are found using the shooting method coupled with Runge-Kutta procedure (RK-6) and analytical results with the homotopy perturbation method (HPM). A remarkable improvement has been noticed, giving results which agree almost 99.9 % with the numerical results. The results are presented in terms of tables. Recently, Asghar et al. [6] used Adomian decomposition method for solving the governing problem.

In this study, we will use HPM for flow in a slowly deforming channel with weak permeability. HPM was first proposed by the Chinese mathematician Ji-Huan He [7 – 10]. The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of 'deformations', the solution for each of which is 'close' to that at the previous stage of 'deformation'. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of 'deformation' gives the desired solution. One of the most remarkable features of the HPM is that usually just a few perturbation terms are sufficient for obtaining a reasonably accurate solution. This technique has been employed to solve a large variety of linear and nonlinear problems [11 – 27]. The interested reader can see the References [28 – 38] for last development of HPM.

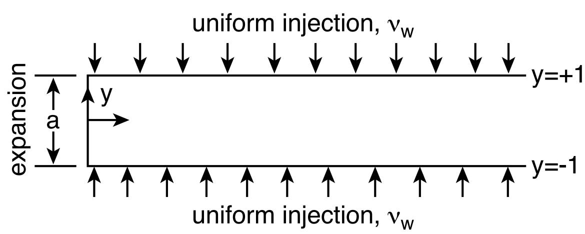

Fig. 1. Flow Geometry.

2. Statement of the Problem

Let us consider a rectangular channel with large aspect ratio of width to height which then can be assumed as infinite. One end of the channel is closed by a membrane which is stretched (shrunk) as the channel is expanded (contracted). The coordinate system considered in the present study is shown in Figure 1. With x indicating the axial direction and y the normal direction, the corresponding axial and transverse velocity components are defined as u and v, respectively. The equations governing the unsteady two-dimensional channel flow problem are

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right], \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right], \tag{3}$$

in which ρ, p, ν, and t are dimensional density, pressure, kinematic viscosity, and time, respectively. The channel walls regress or contract in the normal direction and thus the separation is a function of time. Accordingly, the porosity of the walls allows fluid injection or suction in the direction perpendicular to the walls. The injection/suction absolute velocity v_w is positive/negative, respectively. The axial velocity is zero at $x=0$. (1) – (3) are subject to the following boundary conditions:

$$u=0, \quad v=-v_w \text{ at } y=a(t), \tag{4}$$

$$\frac{\partial u}{\partial y}=0, \quad v=0 \text{ at } y=0, \tag{5}$$

$$u=0 \text{ at } x=0. \tag{6}$$

Introducing the stream function ψ and vorticity ζ via

$$u=\frac{\partial \psi}{\partial y}, \quad v=-\frac{\partial \psi}{\partial x}, \tag{7}$$

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \tag{8}$$

the continuity equation is identically satisfied and the momentum equation reduces to

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}=\nu\left[\frac{\partial^2 \zeta}{\partial x^2}+\frac{\partial^2 \zeta}{\partial y^2}\right]. \tag{9}$$

The problem allows a similar solution:

$$\psi=\nu x G(a) F(\eta,t), \tag{10}$$

where $\eta=y/a$ and $F(\eta,t)$ is independent of the axial coordinates. From (7) and (10), the axial and normal velocities are

$$u=\frac{\nu x}{a}G(a)F_\eta, \quad v=-\nu G(a)F(\eta,t), \tag{11}$$

in which $F_\eta=\frac{\partial F}{\partial \eta}$. At this point, one may realize that $u_{xx}=v_x=0$ and revisit the vorticity equation. In order to ensure dimensional homogeneity we write $G(a)=1/a$; the axial and transverse velocities can finally be written as

$$u=\frac{\nu x}{a^2}F_\eta, \quad v=-\frac{\nu}{a}F(\eta,t). \tag{12}$$

Noting that the normal velocity v is independent of x, the vorticity equation reduces to $\zeta=-\frac{\partial u}{\partial y}$. Likewise (3) becomes $p_{yx}=0$. Upon substitution of ζ and insertion of (1) into the vorticity transport equation (9), one obtains

$$u_{yt}+uu_{yx}+vu_{yy}=\nu u_{yyy}. \tag{13}$$

Insertion of (12) into (13) yields

$$\Big[F_{\eta\eta\eta}+FF_{\eta\eta}+F_\eta(2\alpha-F_\eta) +2\alpha\eta F_{\eta\eta}-\frac{\alpha^2}{\nu}F_{\eta t}\Big]_\eta=0, \tag{14}$$

where α is the wall expansion ratio defined by

$$\alpha(t)=\dot{a}a/\nu. \tag{15}$$

Note that allowing $p_{xy}=0$ in (2) will also lend itself as a method for arriving at (14). The boundary conditions given by (4) – (6) can now be updated to account for the normalization. This in turn gives

$$F_{\eta\eta}=0, \quad F=0 \text{ at } \eta=0, \tag{16}$$

$$F_\eta=0, \quad F=A\alpha=R \text{ at } \eta=1. \tag{17}$$

A similar solution with respect to space and time can now be developed. If our function F is made dependent on η and $\alpha(t)$ instead of (η,t), then one obtains $F_{\eta t}=0$ by setting α to be constant or quasi-constant in time. In that event, the value of the expansion ratio α can be specified by its initial value, namely

$$\alpha=\dot{a}a/\nu=\dot{a}_0 a_0/\nu, \tag{18}$$

where a_0 and $\dot{a}_0=\mathrm{d}a_0/\mathrm{d}t$ denote the initial channel height and channel expansion ratio, respectively. By integrating (18) with respect to time, a similar solution for the temporal channel height evolution can be obtained. This is given by

$$a(t)/a_0=\sqrt{1+2\nu\alpha t a_0{}^{-2}}. \tag{19}$$

For a pyhsical setting in which the injection coefficient A is constant (e. g. propellant burning), an expression for the time-dependent injection velocity evolution can be deduced. One gets

$$\dot{a}/\dot{a}_0 = v_w(t)/v_w(0) = (1 + 2v\alpha t a_0{}^{-2})^{-\frac{1}{2}}. \quad (20)$$

Under the foregoing self-similarity conditions, an ordinary differential equation (ODE) for the principal function F can be derived by direct integration of (14), which is as follows:

$$F''' + FF'' + F'(2\alpha - F') + \alpha\eta F'' = K, \quad (21)$$

where a prime replaces $\mathrm{d}/\mathrm{d}\eta$ and K is the constant of integration. The boundary conditions now turn out to be of the following form:

$$F'' = 0, \quad F = 0 \text{ at } \eta = 0, \quad (22)$$

$$F' = 0, \quad F = R \text{ at } \eta = 1. \quad (23)$$

In the next section we will solve the problem consisting of (21) and the boundary conditions (22) and (23).

3. Solution by the Homotopy Perturbation Method

In order to solve (21) – (23) by the homotopy perturbation method, we construct the following homotopy:

$$\begin{aligned} F'''(\eta) = p\big[&K - 2\alpha F'(\eta) - \alpha\eta F''(\eta) \\ &+ (F'(\eta))^2 - F(\eta)F''(\eta)\big]. \end{aligned} \quad (24)$$

Assume the solution of (21) be in the form

$$F = F_0 + pF_1 + p^2F_2 + p^3F_3 + \ldots. \quad (25)$$

Substituting (25) into (24) and collecting terms of the same power of p, we get the following set of differential equations:

$$\begin{aligned} p^0 &: F_0{}''' = 0, \\ p^1 &: F_1{}''' = K - 2\alpha F_0{}' + (F_0{}')^2 - \alpha\eta F_0{}'' - F_0F_0{}'', \\ p^2 &: F_2{}''' = -2\alpha F_1{}' + (2F_0{}'F_1{}') \\ &\qquad - \alpha\eta F_1{}'' - (F_0F_1{}'' + F_1F_0{}''), \\ p^3 &: F_3{}''' = -2\alpha F_2{}' + (2F_0{}'F_2{}' + (F_1{}')^2) \\ &\qquad - \alpha\eta F_2{}'' - (F_0F_2{}'' + F_1F_1{}'' + F_2F_0{}''), \\ p^4 &: F_4{}''' = -2\alpha F_3{}' + (2F_0{}'F_3{}' + 2F_1{}'F_2{}') - \alpha\eta F_3{}'' \\ &\qquad - (F_0F_3{}'' + F_1F_2{}'' + F_2F_1{}'' + F_3F_0{}''), \\ &\vdots \end{aligned} \quad (26)$$

From (26) we obtain the following iterative formula:

$$F_0{}''' = 0, \quad (27)$$

$$F_1{}''' = K - 2\alpha F_0{}' + A_0 - \alpha\eta F_0{}'' - B_0{}'', \quad (28)$$

$$F_{k+1}{}''' = 2\alpha F_k{}' + A_k - \alpha\eta F_k{}'' - B_k, \quad k \ge 1, \quad (29)$$

where

$$A_k = \sum_{i=0}^{k} F_i{}'F'_{k-i} \text{ and } B_k = \sum_{i=0}^{k} F_iF''_{k-i}. \quad (30)$$

We can solve the above ordinary differential equations, where $F(0) = 0$, $F''(0) = 0$, and we assume an additional condition $F'(0) = \beta$, where β and K will be determined using the boundary conditions at $\eta = 1$, i. e., $F(1) = R$ and $F'(1) = 0$. Thus, we now successively obtain

$$\begin{aligned} F_0 &= \beta\eta, \\ F_1(\eta) &= \frac{1}{6}(K - 2\alpha\beta + \beta^2)\eta^3, \\ F_2(\eta) &= -\frac{1}{30}(K\alpha - 2\alpha^2\beta + \alpha\beta^2)\eta^5, \\ F_3(\eta) &= \frac{1}{2520}(12\alpha^2K + K^2 - 24\alpha^3\beta \\ &\qquad + 2K\beta^2 + 8\alpha^2\beta^2 + \beta^4)\eta^7, \\ F_4(\eta) &= -\frac{1}{45360}(2K\beta^3 + 4\beta^4\alpha + \beta^5 + 8K\alpha\beta^2 \\ &\qquad + K^2\beta + 4K\alpha^2\beta + 4K^2\alpha + 24K\alpha^3 - 48\alpha^4\beta)\eta^9, \\ &\vdots \end{aligned}$$

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Then the series solutions expression by HPM can be written in the form

$$\begin{aligned} F(\eta) = &F_0(\eta) + F_1(\eta) + F_2(\eta) \\ &+ F_3(\eta) + F_4(\eta) + \ldots. \end{aligned} \quad (31)$$

4. Results and Discussions

For practical numerical computations, we take finite j-term approximation of $F(\eta)$,

$$F(n) = \sum_{i=0}^{j-1} F_i.$$

The recursive algorithm (27) – (30) is coded in the computer algebra package Maple. To achieve reasonable accuracy we obtain the 21-term approximation of

Table 1. Comparison of HPM solutions with those of Boutros et al. [39] and Majdalani et al. [2] for self-axial velocity at $\alpha = 0.5$ and $R = 5$.

y	Numerical (Shooting method coupled with RK-6)	— Analytical — HPM	Boutros et al.	Majdalani et al.	— Percentage error (%) — HPM	Boutros et al.	Majdalani et al.
0.00	1.559640	1.559474	1.556324	1.536002	0.010649	0.212613	1.515606
0.05	1.554982	1.554822	1.551780	1.531846	0.010312	0.205919	1.487863
0.10	1.541032	1.540888	1.538164	1.519377	0.009337	0.186109	1.405227
0.15	1.517864	1.517743	1.515522	1.498596	0.007964	0.154296	1.269415
0.20	1.485595	1.485503	1.483935	1.469505	0.006192	0.111740	1.083068
0.25	1.444390	1.444331	1.443517	1.432114	0.004113	0.060441	0.849909
0.30	1.394461	1.394435	1.394421	1.386445	0.001865	0.002868	0.574846
0.35	1.336065	1.336071	1.336839	1.332539	0.000477	0.057931	0.263909
0.40	1.269503	1.269540	1.271006	1.270464	0.002950	0.118393	0.075699
0.45	1.195123	1.195188	1.197207	1.200325	0.005417	0.174375	0.435269
0.50	1.113314	1.113403	1.115778	1.122275	0.007954	0.221321	0.804894
0.55	1.024510	1.024617	1.027110	1.036527	0.010406	0.253780	1.172951
0.60	0.929184	0.929302	0.931656	0.943364	0.012739	0.266040	1.526070
0.65	0.827846	0.827971	0.829933	0.843156	0.015093	0.252100	1.849378
0.70	0.721046	0.721170	0.722523	0.736373	0.017173	0.204841	2.125662
0.75	0.609363	0.609480	0.610078	0.623597	0.019213	0.117336	2.335882
0.80	0.493409	0.493513	0.493322	0.505538	0.021171	0.017632	2.458204
0.85	0.373823	0.373909	0.373046	0.383052	0.023052	0.207852	2.468815
0.90	0.251268	0.251330	0.250109	0.257149	0.024832	0.461260	2.340529
0.95	0.126428	0.126461	0.125435	0.129010	0.025826	0.785427	2.042269
1.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Table 2. Comparison of HPM solutions with those of Boutros et al. [39] and Majdalani et al. [2] for self-axial velocity at $\alpha = -0.5$ and $R = 5$.

y	Numerical (Shooting method coupled with RK-6)	— Analytical — HPM	Boutros et al.	Majdalani et al.	— Percentage error (%) — HPM	Boutros et al.	Majdalani et al.
0.00	1.492361	1.492595	1.515104	1.535426	0.015687	1.523961	2.885696
0.05	1.488855	1.489084	1.511345	1.531279	0.015388	1.510557	2.849438
0.10	1.478318	1.478534	1.500051	1.518837	0.014595	1.470117	2.740885
0.15	1.460699	1.460893	1.481177	1.498104	0.013288	1.401932	2.560760
0.20	1.435914	1.436081	1.454653	1.469083	0.011622	1.305022	2.309957
0.25	1.403855	1.403991	1.420383	1.431787	0.009678	1.177330	1.989664
0.30	1.364397	1.364499	1.378261	1.386238	0.007461	1.016127	1.600780
0.35	1.317404	1.317471	1.328174	1.332474	0.005067	0.817517	1.143916
0.40	1.262274	1.262772	1.270018	1.270560	0.039487	0.613496	0.656434
0.45	1.200280	1.200279	1.203708	1.200590	0.000089	0.285600	0.025827
0.50	1.129917	1.129885	1.129200	1.122702	0.002860	0.063456	0.638542
0.55	1.051573	1.051513	1.046507	1.037090	0.005670	0.481754	1.377270
0.60	0.965209	0.965127	0.955722	0.944014	0.008473	0.982896	2.195897
0.65	0.870834	0.870736	0.857047	0.843823	0.011272	1.583195	3.101739
0.70	0.768513	0.768404	0.750818	0.736968	0.014139	2.302498	4.104680
0.75	0.658372	0.658261	0.637541	0.634023	0.016914	3.164017	3.698365
0.80	0.540609	0.540502	0.517928	0.505712	0.019783	4.195454	6.455127
0.85	0.415494	0.415401	0.392938	0.385530	0.022367	5.428719	7.211656
0.90	0.283381	0.283311	0.263822	0.259708	0.024872	6.902015	8.353771
0.95	0.144708	0.144668	0.132178	0.131204	0.027608	8.658816	9.331896
1.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

$F(\eta)$, i. e. $F(\eta) = \sum_{i=0}^{20} F_i$, where the first five terms are given above.

The unknown constants β and K can be determined by using the boundary conditions (22) and then substituting these values into (31) to obtain the final solution of the problem. Table 1 and Table 2 indicate the comparison of HPM solutions with those of Boutros et al. [39] and Majdalani et al. [2] for self-axial velocity at $\alpha = \pm 0.5$ and $R = 5$ and show the percentage error. Also Table 1 and Table 2 show that the analytic results

(HPM) obtained agree very well with the numerical results (shooting method coupled with RK-6) for a good range of these parameters.

5. Second Adaptation of Homotopy Perturbation Method

In this section, we will use a second adaptation of HPM which is described in [36, 37].

5.1. Qualitative Sketch/Trial Function Solution

From physical understandings and boundary conditions, we choose such initial guess

$$F'(\eta) = \beta(1-\eta)\mathrm{e}^{-a\eta}, \tag{32}$$

where a is an unknown constant to be further identified. It is obvious that (32) satisfies the boundary conditions

$$F'(0) = \beta \quad \text{and} \quad F'(1) = 0.$$

5.2. Construction of Homotopy Equations

According to the initial guess (32), the homotopy equations can be constructed as follows:

$$\begin{aligned} F'''(\eta) - a^2F'(\eta) = p\big[&K - 2\alpha F'(\eta) - \alpha\eta F''(\eta) \\ &+ (F'(\eta))^2 - F(\eta)F''(\eta) - a^2F'(\eta)\big]. \end{aligned} \tag{33}$$

When $p = 0$, we can obtain the initial guess; when $p = 1$, (33) turns out to be the original one.

5.3. Solution Procedure Similar to that of Classical Perturbation Methods

We can use the homotopy parameter p as an expanding parameter used in the classic perturbation methods. The simplest way is the method of straightforward expansion:

$$F = F_0 + pF_1 + p^2F_2 + p^3F_3 + \dots . \tag{34}$$

Generally, we stop before the second iteration. Setting $p = 1$, we obtain the first-order approximation which reads

$$F(\eta) = F_0(\eta) + F_1(\eta). \tag{35}$$

5.4. Optimal Identification of the Unknown Parameter in the Trial Functions

There are many approaches to identify the unknown parameters in the obtained approximation, for example, the method of weighted residuals, especially the least squares method. For the present problem, we set

$$\begin{aligned} \int_0^1 \beta(1-\eta)\mathrm{e}^{-a\eta}\big\{&F''' + FF'' + F'(2\alpha - F') \\ &+ \alpha\eta F'' - K\big\}\mathrm{d}\eta \end{aligned} \tag{36}$$

and

$$F(0) = 0 \tag{37}$$

to identify the unknown constant a. Ariel's identification of the unknown constant is equivalent to (36).

6. Conclusion

In this study, HPM was employed to obtain the analytical solution for flow in a slowly deforming channel with weak permeability. We obtained the approximate analytical solution of the equation in the form of a convergent power series with easily computable components. The present work shows 99.9 % accuracy of agreement with the numerical results. The method is extremely simple, easy to use, and is very accurate for solving nonlinear equations. It is shown that HPM is a very fast convergent, precise, and cost efficient tool for solving nonlinear problems.

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